

Correlated fractional counting processes on a finite time interval

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Abstract

We present some correlated fractional counting processes on a finite time interval. This will be done by considering a slight generalization of the processes in [9]. The main case concerns a class of space-time fractional Poisson processes and, when the correlation parameter is equal to zero, the univariate distributions coincide with the ones of the space-time fractional Poisson process in [24]. On the other hand, when we consider the time fractional Poisson process, the multivariate finite dimensional distributions are different from the ones presented for the renewal process in [26]. Another case concerns a class of fractional negative binomial processes.

Keywords: Poisson process, negative binomial process, weighted process.

Mathematical Subject Classification: 60G22, 60G55, 60E05, 33E12.

1 Introduction

Several fractional processes in the literature are defined by considering some known equations in terms of suitable fractional derivatives. In this paper we are interested in particular Lévy counting processes, as in the recent paper [5]; in particular we deal with Poisson and negative binomial processes. There is a wide literature on fractional Poisson processes: see e.g. [16], [19], [7], [8], [24] and [26] (we also cite [15] and [20] where their representation in terms of randomly time-changed and subordinated processes was studied in detail). Some references with fractional negative binomial processes are [5] (see Example 3) and [28]. Among the other fractional processes in the literature we recall the diffusive processes (see e.g. [2], [3], [18] [22], [27]), the telegraph processes in [21] and the pure birth processes in [23].

Often the results for these fractional processes are given in terms of the Mittag-Leffler function

$$E_{\alpha,\beta}(x) := \sum_{r \geq 0} \frac{x^r}{\Gamma(\alpha r + \beta)}$$

(see e.g. [25], page 17); we also recalled the generalized Mittag-Leffler function

$$E_{\alpha,\beta}^{\gamma}(x) := \sum_{r \geq 0} \frac{(\gamma)^{(r)} x^r}{r! \Gamma(\alpha r + \beta)},$$

where

$$(\gamma)^{(r)} := \begin{cases} \gamma(\gamma+1) \cdots (\gamma+r-1) & \text{if } r \geq 1 \\ 1 & \text{if } r = 0 \end{cases} \quad (\text{for } \gamma \in \mathbb{R})$$

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is the rising factorial (also called Pochhammer symbol), and $E_{\alpha,\beta}^\gamma$ coincides with $E_{\alpha,\beta}$ when $\gamma = 1$.

In this paper we consider some processes $\{N_\rho(\cdot) : \rho \in [0, 1]\}$ on a finite time interval $[0, T]$, for some $T \in (0, \infty)$. More precisely $N_\rho(\cdot) = \{N_\rho(t) : t \in [0, T]\}$ is defined by

$$N_\rho(t) := \sum_{n=1}^{M_g} 1_{[0,t]}(X_n^{F,\rho}),$$

where M_g is a nonnegative integer valued random variable with probability generating function g , i.e.

$$g(u) := \mathbb{E} [u^{M_g}],$$

and $\{X_n^{F,\rho} : n \geq 1\}$ is a sequence of random variables with (common) distribution function F such that $F(0) = 0$ and $F(T) = 1$, and independent of M_g ; moreover the correlation coefficient between any pair of random variables X_n and X_m , with $n \neq m$, is equal to a common value $\rho \in [0, 1]$.

Remark 1. We have $N_\rho(T) = M_g$; thus the distribution of $N_\rho(T)$ does not depend on ρ .

In this way we are considering a slight generalization of the processes presented in [9]; indeed we can recover several formulas in [9] by setting $g(u) = e^{\lambda(u-1)}$ for some $\lambda > 0$ (which concerns a Poisson distributed random variable with mean λ), and $F(t) = t$ for $t \in [0, 1]$, where $T = 1$. The case without correlation, i.e. the case $\rho = 0$, appears in [4]; see also [17] where that process is considered as a claim number process in insurance. Here, in view of what follows, we recall the following formulas (see e.g. (9) and (10) in [9]): we have the probability generating function

$$G_{N_\rho(t)}(u) = \rho(1 - F(t)) + \rho F(t)g(u) + (1 - \rho)g(1 - F(t) + F(t)u), \quad (1)$$

and the probability mass function

$$P(N_\rho(t) = k) = (1 - \rho)P(N_0(t) = k) + \rho\{(1 - F(t))1_{k=0} + F(t)P(M_g = k)\} \text{ (for all } k \geq 0), \quad (2)$$

where

$$P(N_0(t) = k) = \sum_{n=k}^{\infty} \binom{n}{k} F^k(t)(1 - F(t))^{n-k} P(M_g = n) \text{ (for all } k \geq 0) \quad (3)$$

concerns the case $\rho = 0$ (see (2.4) in [4]).

As pointed out in [4], this class of counting processes can be useful to tackle the problem of overdispersion and underdispersion in the analysis of count data where correlations between events are present. A possible application can be given for example in models of non-exponential extinction of radiation in correlated random media (see e.g. [13]). We also remark that, as far as the the marginal distribution of each random variable $N_\rho(t)$, in (2) we have a mixture between three probability mass functions, i.e. $\{P(N_0 = k) : k \geq 0\}$, $\{1_{k=0} : k \geq 0\}$ and $\{P(M_g = k) : k \geq 0\}$, and the weights are $1 - \rho$, $\rho(1 - F(t))$ and $\rho F(t)$, respectively.

The aim of this paper is to present some correlated fractional counting processes by choosing in a suitable way the probability generating function g and a distribution function F above. In Section 2 we present a class of space-time fractional Poisson processes (in fact we have the same univariate distributions of the space-time fractional Poisson process in [24] when $\rho = 0$). A class of fractional negative binomial processes is presented in Section 3.

Finally, since the presentation of the results in [9] refers to the concept of weighted Poisson processes (see also the previous reference [4] concerning the case $\rho = 0$), in the final Section 4 we give some minor results on weighted processes. Even though this section seems to be disconnected from the other ones in this paper, in our opinion it is a small nice enrichment of the content of [9].

2 A class of correlated fractional Poisson processes

For the aims of this section, some preliminaries are needed. Firstly we consider the Caputo (left fractional) derivative $\frac{d^\nu}{dt^\nu}$ of order $\nu > 0$ (see e.g. ${}^C D_{a+}^\nu$ in (2.4.14) and (2.4.15) in [12] with $a = 0$; we use the notation $[x] := \max\{k \in \mathbb{Z} : k \leq x\}$) defined by

$$\frac{d^\nu}{dt^\nu} f(t) := \begin{cases} \frac{1}{\Gamma(n-\nu)} \int_0^t \frac{1}{(t-s)^{\nu-n+1}} \frac{d^n}{ds^n} f(s) ds & \text{if } \nu \text{ is not integer, where } n = [\nu] + 1 \\ \frac{d^\nu}{dt^\nu} f(t) & \text{if } \nu \text{ is integer} \end{cases} \quad (\text{for all } t \geq 0);$$

note that, since here we consider $\nu \in (0, 1]$, we have (see e.g. (2.4.17) in [12] with $a = 0$)

$$\frac{d^\nu}{dt^\nu} f(t) := \begin{cases} \frac{1}{\Gamma(1-\nu)} \int_0^t \frac{1}{(t-s)^\nu} \frac{d}{ds} f(s) ds & \text{if } \nu \in (0, 1) \\ \frac{d}{dt} f(t) & \text{if } \nu = 1 \end{cases} \quad (\text{for all } t \geq 0).$$

We also consider the (fractional) difference operator $(I - B)^\alpha$ in [24]; more precisely I is the identity operator, B is the backward shift operator defined by $Bf(k) = f(k-1)$ and $B^{r-1}Bf(k) = f(k-r)$, and therefore

$$(I - B)^\alpha = \sum_{j=0}^{\infty} (-1)^j \binom{\alpha}{j} B^j. \quad (4)$$

We now recall that Orsingher and Polito in [24] considered the space-time fractional Poisson process $\{N_0^{\alpha,\nu}(t) : t \geq 0\}$, for $\alpha, \nu \in (0, 1]$, whose probability mass functions $\{p_k(t) : k \geq 0\}$ solve the Cauchy problem

$$\begin{cases} \frac{d^\nu}{dt^\nu} p_k(t) = -\lambda^\alpha (I - B)^\alpha p_k(t) \\ p_k(0) = \begin{cases} 0, & k > 0, \\ 1, & k = 0. \end{cases} \end{cases}$$

The explicit form of the probability generating function of this process has the form (see [24], equation (2.28))

$$\mathbb{E} \left[u^{N_0^{\alpha,\nu}(t)} \right] = E_{\nu,1}(-\lambda^\alpha t^\nu (1-u)^\alpha).$$

In this section we consider a class of correlated space-time fractional Poisson processes on a finite time interval $[0, T]$. For $\alpha, \nu \in (0, 1]$ we consider $N_\rho(\cdot) = N_\rho^{\alpha,\nu}(\cdot)$ such that the probability generating function of M_g is

$$g(u) := E_{\nu,1}(-\lambda^\alpha T^\nu (1-u)^\alpha), \quad (5)$$

and the distribution function of the random variables $\{X_n^{F,\rho} : n \geq 1\}$ is

$$F(t) := (t/T)^{\nu/\alpha} \quad (\text{for } t \in [0, T]).$$

In what follows we present the probability generating functions in Proposition 2.1 and the corresponding probability mass functions in Proposition 2.2. Moreover, in Proposition 2.3, we give an equation for the probability mass functions in Proposition 2.2 with respect to time t .

Proposition 2.1. *The probability generating functions $\{G_{N_\rho^{\alpha,\nu}(t)} : t \in [0, T]\}$ are*

$$\begin{aligned} G_{N_\rho^{\alpha,\nu}(t)}(u) &= \rho(1 - (t/T)^{\nu/\alpha}) + \rho(t/T)^{\nu/\alpha} E_{\nu,1}(-\lambda^\alpha T^\nu (1-u)^\alpha) \\ &\quad + (1 - \rho) E_{\nu,1}(-\lambda^\alpha t^\nu (1-u)^\alpha). \end{aligned}$$

Proof. We have

$$\begin{aligned} G_{N_\rho^{\alpha,\nu}(t)}(u) &= \rho(1 - (t/T)^{\nu/\alpha}) + \rho(t/T)^{\nu/\alpha} E_{\nu,1}(-\lambda^\alpha T^\nu (1-u)^\alpha) \\ &\quad + (1 - \rho) E_{\nu,1}(-\lambda^\alpha T^\nu (1 - \{1 - (t/T)^{\nu/\alpha} + (t/T)^{\nu/\alpha} u\})^\alpha) \end{aligned}$$

by (1), and we conclude with some manipulations of the last term. \square

Remark 2. By Proposition 2.1, if $\rho = 0$ we have the probability generating function

$$G_{N_0^{\alpha,\nu}(t)}(u) = E_{\nu,1}(-\lambda^\alpha t^\nu (1-u)^\alpha) \quad (6)$$

which coincides with the one presented in the last case of Table 1 in [24]; note that (6) is a generalization of (5) with $t \in [0, T]$ instead of $t = T$. Thus the univariate distributions of the random variables $\{N_0^{1,\nu}(t) : t \in [0, T]\}$ (for the case $\alpha = 1$) coincide with the ones of the random variables of the renewal process $\{M(t) : t \in [0, T]\}$ in [26] (restricted to the same finite time interval). On the other hand one can check that the multivariate finite dimensional marginal distributions are different from the ones in [26] (and, in fact, $\{N_\rho^{\alpha,\nu}(t) : t \in [0, T]\}$ is not a renewal process). We explain this with a simple example where we take into account that

$$P(M(s) = 1) = P(N_0^{1,\nu}(s) = 1) = \lambda s^\nu E_{\nu,\nu+1}^2(-\lambda s^\nu) \text{ (for } s \in [0, T])$$

by (2.5) in [8]. In fact, for $t \in (0, T)$, we have

$$P(M(t) = 1, M(T) = 1) = \lambda t^\nu E_{\nu,\nu+1}^2(-\lambda t^\nu) E_{\nu,1}(-\lambda(T-t)^\nu) \quad (7)$$

by combining (11) and (14) in [26] (with $(t_1, t_2) = (t, T)$ and $(n_1, n_2) = (1, 1)$) with (2) and (4) in the same reference, and

$$P(N_0^{1,\nu}(t) = 1, N_0^{1,\nu}(T) = 1) = \frac{t}{T} \lambda T^\nu E_{\nu,\nu+1}^2(-\lambda T^\nu) \quad (8)$$

because $P(N_0^{\alpha,\nu}(t) = 1 | N_0^{\alpha,\nu}(T) = 1) = \frac{t}{T}$ by construction. Then (7) and (8) coincide only for the non-fractional case $\nu = 1$ (see Figure 1 below).

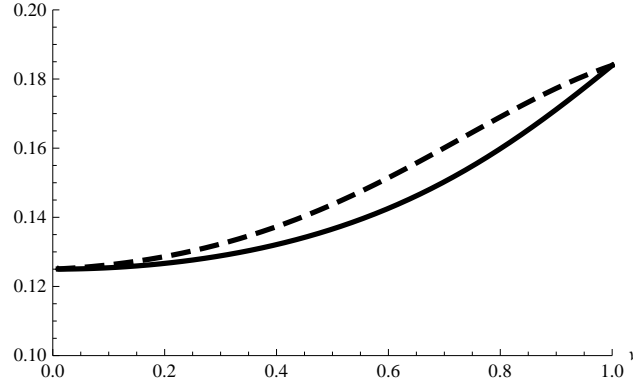


Figure 1: The probabilities (7) (dashed line) and (8) (solid line) versus $\nu \in (0, 1]$ for $t = 1/2$ and $T = \lambda = 1$.

Proposition 2.2. The probability mass functions $\{P(N_\rho^{\alpha,\nu}(t) = \cdot) : t \in [0, T]\}$ are

$$\begin{aligned} P(N_\rho^{\alpha,\nu}(t) = k) = & (1-\rho) \frac{(-1)^k}{k!} \sum_{r=0}^{\infty} \frac{(-\lambda^\alpha t^\nu)^r}{\Gamma(\nu r + 1)} \frac{\Gamma(\alpha r + 1)}{\Gamma(\alpha r + 1 - k)} + \rho \left\{ \left(1 - (t/T)^{\frac{\nu}{\alpha}}\right) 1_{k=0} \right. \\ & \left. + (t/T)^{\frac{\nu}{\alpha}} \cdot \frac{(-1)^k}{k!} \sum_{r=0}^{\infty} \frac{(-\lambda^\alpha T^\nu)^r}{\Gamma(\nu r + 1)} \frac{\Gamma(\alpha r + 1)}{\Gamma(\alpha r + 1 - k)} \right\} \text{ (for all } k \geq 0). \end{aligned}$$

Proof. Firstly we have

$$P(M_g = n) = P(N_\rho^{\alpha, \nu}(T) = k) = \frac{(-1)^n}{n!} \sum_{r=0}^{\infty} \frac{(-\lambda^\alpha T^\nu)^r}{\Gamma(\nu r + 1)} \frac{\Gamma(\alpha r + 1)}{\Gamma(\alpha r + 1 - n)} \quad (\text{for all } n \geq 0) \quad (9)$$

by the probability generating function in (5) (see (1.8) in [24]) and by Remark 1. Moreover, if we consider (3), we get

$$\begin{aligned} P(N_0^{\alpha, \nu}(t) = k) &= \sum_{n=k}^{\infty} \binom{n}{k} (t/T)^{\frac{\nu}{\alpha} k} \left(1 - (t/T)^{\frac{\nu}{\alpha}}\right)^{n-k} \frac{(-1)^n}{n!} \sum_{r=0}^{\infty} \frac{(-\lambda^\alpha T^\nu)^r}{\Gamma(\nu r + 1)} \frac{\Gamma(\alpha r + 1)}{\Gamma(\alpha r + 1 - n)} \\ &= \frac{(-1)^k}{k!} (t/T)^{\frac{\nu}{\alpha} k} \sum_{n=k}^{\infty} \frac{(-1)^{n-k}}{(n-k)!} \left(1 - (t/T)^{\frac{\nu}{\alpha}}\right)^{n-k} \\ &\quad \cdot \sum_{r=0}^{\infty} \frac{(-\lambda^\alpha t^\nu)^r}{\Gamma(\nu r + 1)} (T/t)^{\nu r} \frac{\Gamma(\alpha r + 1)}{\Gamma(\alpha r + 1 - k)} \frac{\Gamma(\alpha r + 1 - k)}{\Gamma(\alpha r + 1 - n)} \\ &= \frac{(-1)^k}{k!} (t/T)^{\frac{\nu}{\alpha} k} \sum_{r=0}^{\infty} \frac{(-\lambda^\alpha t^\nu)^r}{\Gamma(\nu r + 1)} (T/t)^{\nu r} \frac{\Gamma(\alpha r + 1)}{\Gamma(\alpha r + 1 - k)} \\ &\quad \cdot \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} \left(1 - (t/T)^{\frac{\nu}{\alpha}}\right)^j \frac{\Gamma(\alpha r + 1 - k)}{\Gamma(\alpha r + 1 - k - j)} \quad (\text{for all } k \geq 0). \end{aligned}$$

Then, by a well-known ‘‘Newton’s generalized binomial theorem’’, we obtain

$$\begin{aligned} P(N_0^{\alpha, \nu}(t) = k) &= \frac{(-1)^k}{k!} (t/T)^{\frac{\nu}{\alpha} k} \sum_{r=0}^{\infty} \frac{(-\lambda^\alpha t^\nu)^r}{\Gamma(\nu r + 1)} (T/t)^{\nu r} \frac{\Gamma(\alpha r + 1)}{\Gamma(\alpha r + 1 - k)} \left(1 + (t/T)^{\frac{\nu}{\alpha}} - 1\right)^{\alpha r - k} \\ &= \frac{(-1)^k}{k!} \sum_{r=0}^{\infty} \frac{(-\lambda^\alpha t^\nu)^r}{\Gamma(\nu r + 1)} (t/T)^{\frac{\nu}{\alpha} k - \nu r + \nu r - \frac{\nu}{\alpha} k} \frac{\Gamma(\alpha r + 1)}{\Gamma(\alpha r + 1 - k)} \\ &= \frac{(-1)^k}{k!} \sum_{r=0}^{\infty} \frac{(-\lambda^\alpha t^\nu)^r}{\Gamma(\nu r + 1)} \frac{\Gamma(\alpha r + 1)}{\Gamma(\alpha r + 1 - k)} \quad (\text{for all } k \geq 0) \end{aligned}$$

where, as we expected by (6), $P(N_0^{\alpha, \nu}(t) = k)$ here meets $P(N_\rho^{\alpha, \nu}(T) = k)$ in (9) (here we have t and k in place of T and n in (9)). We conclude the proof by considering (2) together with the last above expression obtained for the case $\rho = 0$. \square

In view of the next Proposition 2.3 we remark that in a part of the proof we refer to Theorem 2 in [6] which can be derived by referring to a subordinated representation of the space-time fractional Poisson process in terms of both stable subordinator and its inverse (see also (3.20), together with (3.1), in the same reference).

Proposition 2.3. *Let $\{P(N_\rho^{\alpha, \nu}(t) = \cdot) : t \in [0, T]\}$ be the probability mass functions in Proposition 2.2. Then we have the following equations: for $k = 0$,*

$$\begin{aligned} \frac{d^\nu}{dt^\nu} P(N_\rho^{\alpha, \nu}(t) = 0) &= -\lambda^\alpha P(N_\rho^{\alpha, \nu}(t) = 0) + \lambda^\alpha \rho \\ &\quad - \rho (t/T)^{\nu/\alpha} \left(\lambda^\alpha + t^{-\nu} \frac{\Gamma(\nu/\alpha + 1)}{\Gamma(\nu/\alpha - \nu + 1)} \right) (1 - P(N_\rho^{\alpha, \nu}(T) = 0)); \end{aligned}$$

for all $k \geq 1$,

$$\begin{aligned} \frac{d^\nu}{dt^\nu} P(N_\rho^{\alpha, \nu}(t) = k) &= -\lambda^\alpha (I - B)^\alpha P(N_\rho^{\alpha, \nu}(t) = k) + \lambda^\alpha \rho (1 - (t/T)^{\nu/\alpha}) (-1)^k \binom{\alpha}{k} \\ &\quad + \rho (t/T)^{\nu/\alpha} \left[\lambda^\alpha (I - B)^\alpha + t^{-\nu} \frac{\Gamma(\nu/\alpha + 1)}{\Gamma(\nu/\alpha - \nu + 1)} \right] P(N_\rho^{\alpha, \nu}(T) = k). \end{aligned}$$

In all cases we have the initial conditions $P(N_\rho^{\alpha,\nu}(0) = 0) = 1$ and $P(N_\rho^{\alpha,\nu}(0) = k) = 0$ for all $k \geq 1$.

Proof. The initial conditions trivially hold. Throughout this proof we consider the notation

$$p_k^{\alpha,\nu}(t) = P(N_0^{\alpha,\nu}(t) = k) \text{ (for all } k \geq 0)$$

for the probability mass function concerning the case $\rho = 0$. Then, by (2) and Remark 1, we get

$$\begin{aligned} \frac{d^\nu}{dt^\nu} P(N_\rho^{\alpha,\nu}(t) = k) &= (1 - \rho) \frac{d^\nu}{dt^\nu} p_k^{\alpha,\nu}(t) + \rho \left\{ -\frac{1}{T^{\nu/\alpha}} 1_{k=0} \frac{d^\nu}{dt^\nu} t^{\nu/\alpha} + \frac{1}{T^{\nu/\alpha}} P(N_\rho^{\alpha,\nu}(T) = k) \frac{d^\nu}{dt^\nu} t^{\nu/\alpha} \right\} \\ &= (1 - \rho) \frac{d^\nu}{dt^\nu} p_k^{\alpha,\nu}(t) - \frac{\rho}{T^{\nu/\alpha}} \{1_{k=0} - P(N_\rho^{\alpha,\nu}(T) = k)\} \frac{d^\nu}{dt^\nu} t^{\nu/\alpha}. \end{aligned}$$

Moreover we have

$$\frac{d^\nu}{dt^\nu} p_k^{\alpha,\nu}(t) = -\lambda^\alpha (I - B)^\alpha p_k^{\alpha,\nu}(t)$$

by Theorem 2 in [6] and

$$\frac{d^\nu}{dt^\nu} t^{\nu/\alpha} = t^{\nu/\alpha - \nu} \frac{\Gamma(\nu/\alpha + 1)}{\Gamma(\nu/\alpha - \nu + 1)}$$

(see e.g. (2.2.11) and (2.4.8) in [12], or a correction of (2.4.28) in the same reference); then we get

$$\begin{aligned} \frac{d^\nu}{dt^\nu} P(N_\rho^{\alpha,\nu}(t) = k) &= -\lambda^\alpha (I - B)^\alpha (1 - \rho) p_k^{\alpha,\nu}(t) \\ &\quad - \rho (t/T)^{\nu/\alpha} \{1_{k=0} - P(N_\rho^{\alpha,\nu}(T) = k)\} t^{-\nu} \frac{\Gamma(\nu/\alpha + 1)}{\Gamma(\nu/\alpha - \nu + 1)}. \end{aligned}$$

From now on we consider the cases $k = 0$ and $k \geq 1$ separately.

Case $k = 0$. Firstly we have

$$(I - B)^\alpha p_0^{\alpha,\nu}(t) = \sum_{j=0}^{\infty} (-1)^j \binom{\alpha}{j} p_{0-j}^{\alpha,\nu}(t) = p_0^{\alpha,\nu}(t)$$

by (4); therefore

$$\frac{d^\nu}{dt^\nu} P(N_\rho^{\alpha,\nu}(t) = 0) = -\lambda^\alpha (1 - \rho) p_0^{\alpha,\nu}(t) - \rho (t/T)^{\nu/\alpha} \{1 - P(N_\rho^{\alpha,\nu}(T) = 0)\} t^{-\nu} \frac{\Gamma(\nu/\alpha + 1)}{\Gamma(\nu/\alpha - \nu + 1)}.$$

then, by (2) and Remark 1,

$$\begin{aligned} \frac{d^\nu}{dt^\nu} P(N_\rho^{\alpha,\nu}(t) = 0) &= -\lambda^\alpha \{P(N_\rho^{\alpha,\nu}(t) = 0) - \rho \{1 - (t/T)^{\nu/\alpha} + (t/T)^{\nu/\alpha} P(N_\rho^{\alpha,\nu}(T) = 0)\}\} \\ &\quad - \rho (t/T)^{\nu/\alpha} \{1 - P(N_\rho^{\alpha,\nu}(T) = 0)\} t^{-\nu} \frac{\Gamma(\nu/\alpha + 1)}{\Gamma(\nu/\alpha - \nu + 1)}. \end{aligned}$$

and, finally, we can check by inspection that the last equation is equivalent to the one in the statement of the proposition.

Case $k \geq 1$. Firstly, again by (2) and Remark 1, we have

$$\begin{aligned} \frac{d^\nu}{dt^\nu} P(N_\rho^{\alpha,\nu}(t) = k) &= -\lambda^\alpha (I - B)^\alpha [P(N_\rho^{\alpha,\nu}(t) = k) \\ &\quad - \rho (1 - (t/T)^{\nu/\alpha}) 1_{k=0} - \rho (t/T)^{\nu/\alpha} P(N_\rho^{\alpha,\nu}(T) = k)] \\ &\quad + \rho (t/T)^{\nu/\alpha} P(N_\rho^{\alpha,\nu}(T) = k) t^{-\nu} \frac{\Gamma(\nu/\alpha + 1)}{\Gamma(\nu/\alpha - \nu + 1)} \\ &= -\lambda^\alpha (I - B)^\alpha P(N_\rho^{\alpha,\nu}(t) = k) + \lambda^\alpha \rho (1 - (t/T)^{\nu/\alpha}) (I - B)^\alpha 1_{k=0} \\ &\quad + \rho (t/T)^{\nu/\alpha} \left[\lambda^\alpha (I - B)^\alpha + t^{-\nu} \frac{\Gamma(\nu/\alpha + 1)}{\Gamma(\nu/\alpha - \nu + 1)} \right] P(N_\rho^{\alpha,\nu}(T) = k); \end{aligned}$$

then we get the desired equation by noting that

$$(I - B)^\alpha 1_{k=0} = \sum_{j=0}^{\infty} (-1)^j \binom{\alpha}{j} 1_{k-j=0} = (-1)^k \binom{\alpha}{k}.$$

The proof is complete. \square

Finally we remark that, even if the equations in Proposition 2.3 have some analogies with other results for fractional Poisson processes in the literature, here some standard techniques do not work because we deal with a finite horizon time case (i.e. $t \in [0, T]$).

3 A class of correlated fractional negative binomial processes

It is well-known that the negative binomial process can be seen as a suitable compound Poisson process with logarithmic distributed summands (see e.g. Proposition 1.1 in [14]). More precisely, for some $p \in (0, 1)$ and some integer $r \geq 1$, we have the probability generating function

$$u \mapsto h^r(m(u)),$$

where:

$$h(u) := e^{\lambda(u-1)}, \text{ with } \lambda = -\log p,$$

is the probability generating function of a Poisson distributed random variable with mean $\lambda = -\log p$;

$$m(u) := \frac{\log(1 - (1-p)u)}{\log p}, \text{ for } |u| < \frac{1}{1-p},$$

is the probability generating function of a logarithmic distributed random variable (obviously we have $m(u) = \infty$ if $|u| \geq \frac{1}{1-p}$).

In this section we present a class of correlated fractional negative binomial processes on a finite time interval $[0, T]$. More precisely we consider the same approach with the probability generating function of a space-time fractional Poisson distributed random variable; thus, for $\alpha, \nu \in (0, 1]$, we have

$$h_{\alpha, \nu}(u) := E_{\nu, 1}(-\lambda^\alpha(1-u)^\alpha)$$

in place of h (note that h coincides with $h_{1, 1}$), again with $\lambda = -\log p$, and this meets g in (5) with $T = 1$. Thus we have

$$g(u) := \left\{ E_{\nu, 1} \left(-(-\log p)^\alpha \left(1 - \frac{\log(1 - (1-p)u)}{\log p} \right)^\alpha \right) \right\}^r = \left\{ E_{\nu, 1} \left(-\log^\alpha \left(\frac{1 - (1-p)u}{p} \right) \right) \right\}^r, \quad (10)$$

where, again, $r \geq 1$ is an integer power of the function $E_{\nu, 1}$, $p \in (0, 1)$ and $|u| < \frac{1}{1-p}$. We remark that g in (10) is the probability generating function of $N_\rho(T)$, but it does not depend on T as happens for g in (5).

As far as the distribution function F is concerned, we argue as in Section 2 as follows: for all $t \in [0, T]$, we want to have the condition

$$G_{N_0^{\alpha, \nu}(t)}(u) = \left\{ E_{\nu, 1} \left(-\log^\alpha \left(\frac{1 - (1-q(t))u}{q(t)} \right) \right) \right\}^r$$

for some $q(\cdot)$ such that $q(t) \in (0, 1]$ for all $t \in [0, T]$ and $q(T) = p$. Then, by (1) with $\rho = 0$ and by (10), we require that

$$\begin{aligned} \frac{1 - (1 - q(t))u}{q(t)} &= \frac{1 - (1 - p)(1 - F(t) + F(t)u)}{p} \\ &= \frac{1 - (1 - p)(1 - F(t)) - (1 - p)F(t)u}{p}; \end{aligned}$$

so, if we divide both numerator and denominator by $1 - (1 - p)(1 - F(t))$, we get

$$q(t) = \frac{p}{1 - (1 - p)(1 - F(t))}.$$

Moreover we have

$$q(t) = \frac{p}{p + (1 - p)F(t)} = \frac{1}{1 + (\frac{1}{p} - 1)F(t)}$$

which yields

$$F(t) := \frac{\frac{1}{q(t)} - 1}{\frac{1}{p} - 1} \quad (\text{for } t \in [0, T]), \quad (11)$$

and the function $q(\cdot)$ has to be a decreasing. We also give a particular example with a choice of $q(\cdot)$, and we provide the corresponding distribution function F .

Example 3.1. *If we set*

$$q(t) = \frac{1 - \lambda}{1 - (1 - \frac{t}{T})\lambda}$$

for some $\lambda \in (0, 1)$, we recover the example in Section 3.3 in [4] (see also Section 4.3 in [9] for a generalization). In fact this choice of $q(\cdot)$ is the analogue of (3.6) in [4]; moreover, if we set $p = 1 - \lambda$, we have

$$q(t) = \frac{p}{1 - (1 - \frac{t}{T})(1 - p)} = \frac{1}{1 + (\frac{1}{p} - 1)\frac{t}{T}}$$

and therefore $F(t) = \frac{t}{T}$.

In what follows we present the probability generating functions in Proposition 3.1 and, for $r = 1$ only, the corresponding probability mass functions in Proposition 3.2 (for $r \geq 2$ we have the r -th convolution of the probability mass function of the case $r = 1$, but we cannot provide manageable formulas). Moreover, in Proposition 3.3, we give an equation for the probability generating functions $\{G_{N_p^{\alpha, \nu}(t)} : t \in [0, T]\}$ in Proposition 3.1 for $r = 1$, $\nu = \alpha$ and $\rho \in \{0, 1\}$; in this case we consider fractional derivatives with respect to their argument u , and not with respect to time t .

Proposition 3.1. *The probability generating functions $\{G_{N_p^{\alpha, \nu}(t)} : t \in [0, T]\}$ are*

$$\begin{aligned} G_{N_p^{\alpha, \nu}(t)}(u) = & \rho \left(1 - \frac{\frac{1}{q(t)} - 1}{\frac{1}{p} - 1} \right) + \rho \frac{\frac{1}{q(t)} - 1}{\frac{1}{p} - 1} \left\{ E_{\nu, 1} \left(-\log^\alpha \left(\frac{1 - (1 - p)u}{p} \right) \right) \right\}^r \\ & + (1 - \rho) \left\{ E_{\nu, 1} \left(-\log^\alpha \left(\frac{1 - (1 - q(t))u}{q(t)} \right) \right) \right\}^r. \end{aligned}$$

Proof. This is an immediate consequence of (1) and the formulas above. \square

In view of the next Proposition 3.2 some preliminaries are needed. Firstly we consider the Stirling numbers $\{s_{k, h} : k \geq h \geq 0\}$; for their definition and some properties used below see e.g. [1], page 824. Moreover

$${}_p\Psi_q \left[\begin{matrix} (a_1, \alpha_1) \dots (a_p, \alpha_p) \\ (b_1, \beta_1) \dots (b_q, \beta_q) \end{matrix} \right] (z) := \sum_{j \geq 0} \frac{\prod_{h=1}^p \Gamma(a_h + \alpha_h j)}{\prod_{k=1}^q \Gamma(b_k + \beta_k j)} \frac{z^j}{j!}$$

is the Fox-Wright function (see e.g. (1.11.14) in [12]) under the convergence condition

$$\sum_{k=1}^q \beta_k - \sum_{h=1}^p \alpha_h > -1 \quad (12)$$

(see e.g. (1.11.15) in [12]).

Proposition 3.2. *If $r = 1$, the probability mass functions $\{P(N_\rho^{\alpha,\nu}(t) = \cdot) : t \in [0, T]\}$ are*

$$P(N_\rho^{\alpha,\nu}(t) = k) = (1 - \rho)P(N_0^{\alpha,\nu}(t) = k) + \rho \left\{ \frac{\frac{1}{p} - \frac{1}{q(t)}}{\frac{1}{p} - 1} 1_{k=0} + \frac{\frac{1}{q(t)} - 1}{\frac{1}{p} - 1} P(N_0^{\alpha,\nu}(T) = k) \right\} \quad (\text{for all } k \geq 0),$$

where, for all $t \in [0, T]$,

$$P(N_0^{\alpha,\nu}(t) = k) = \begin{cases} E_{\nu,1}(-\log^\alpha(1 + A_t)) & \text{if } k = 0 \\ \frac{1}{k!} \frac{(-A_t)^k}{(1+A_t)^k} \sum_{h=1}^k \log^{-h}(1 + A_t) s_{k,h} & \text{if } k \geq 1 \end{cases} \quad (13)$$

and $A_t := \frac{1}{q(t)} - 1$ (note that the convergence condition (12) holds because we have $\alpha + \nu - (\alpha + 1) = \nu - 1 > -1$).

Proof. Firstly we remark that we can only check (13) (concerning the case $\rho = 0$); in fact we obtain the formula for the general case by combining (2), F in (11) and (13). It is well-known that

$$P(N_0^{\alpha,\nu}(t) = k) = \begin{cases} G_{N_0^{\alpha,\nu}(t)}(0) & \text{if } k = 0 \\ \frac{1}{k!} \frac{d^k}{du^k} G_{N_0^{\alpha,\nu}(t)}(u) \Big|_{u=0} & \text{if } k \geq 1. \end{cases} \quad (14)$$

Firstly, if $A_t = \frac{1}{q(t)} - 1$ as in the statement of the proposition, we have

$$G_{N_0^{\alpha,\nu}(t)}(u) = E_{\nu,1} \left(-\log^\alpha \left(\frac{1 - (1 - q(t))u}{q(t)} \right) \right) = E_{\nu,1} (-\log^\alpha(1 + A_t(1 - u))),$$

and we immediately obtain (13) for $k = 0$. Moreover, if we prove that

$$\begin{aligned} & \frac{d^k}{du^k} E_{\nu,1}(-\log^\alpha(1 + A(1 - u))) \\ &= \frac{(-A)^k}{(1 + A(1 - u))^k} \sum_{j \geq 0} \sum_{h=1}^k \frac{(-1)^j \Gamma(\alpha j + 1)}{\Gamma(\alpha j - h + 1) \Gamma(\nu j + 1)} s_{k,h} \log^{\alpha j - h}(1 + A(1 - u)) \end{aligned} \quad (15)$$

$$= \frac{1}{k!} \frac{(-A)^k}{(1 + A)^k} \sum_{h=1}^k \log^{-h}(1 + A) s_{k,h} \cdot {}_2\Psi_2 \left[\begin{matrix} (1, \alpha) & (1, 1) \\ (1 - h, \alpha) & (1, \nu) \end{matrix} \right] (-\log^\alpha(1 + A)); \quad (16)$$

for $k \geq 1$ (and for all $A \in \mathbb{R}$), we obtain (13) for $k \geq 1$ (and the proof is complete) as an immediate consequence of (14) and (16) with $A = A_t$. Therefore in the remaining part of the proof we only prove the first equality (15) by induction; in fact the second equality (16) can be checked by inspection. For $k = 1$ we have

$$\frac{d}{du} E_{\nu,1}(-\log^\alpha(1 + A(1 - u))) = \sum_{j \geq 0} \frac{(-1)^j \alpha j}{\Gamma(\nu j + 1)} \frac{\log^{\alpha j - 1}(1 + A(1 - u))}{1 + A(1 - u)} \cdot (-A),$$

and (15) is proved noting that $s_{1,1} = 1$ and $\alpha j = \frac{\Gamma(\alpha j + 1)}{\Gamma(\alpha j)}$. Now we assume that (15) holds for

$k > 1$. Then we have

$$\begin{aligned}
& \frac{d^{k+1}}{du^{k+1}} E_{\nu,1}(-\log^\alpha(1+A(1-u))) \\
&= \frac{d}{du} \left\{ \frac{(-A)^k}{(1+A(1-u))^k} \sum_{j \geq 0} \sum_{h=1}^k \frac{(-1)^j \Gamma(\alpha j + 1)}{\Gamma(\alpha j - h + 1) \Gamma(\nu j + 1)} s_{k,h} \log^{\alpha j - h}(1+A(1-u)) \right\} \\
&= (-A)^k \left\{ \frac{(-k)(-A)}{(1+A(1-u))^{k+1}} \sum_{j \geq 0} \sum_{h=1}^k \frac{(-1)^j \Gamma(\alpha j + 1)}{\Gamma(\alpha j - h + 1) \Gamma(\nu j + 1)} s_{k,h} \log^{\alpha j - h}(1+A(1-u)) \right. \\
&\quad \left. + \frac{1}{(1+A(1-u))^k} \sum_{j \geq 0} \sum_{h=1}^k \frac{(-1)^j \Gamma(\alpha j + 1)}{\Gamma(\alpha j - h + 1) \Gamma(\nu j + 1)} s_{k,h} \frac{(\alpha j - h) \log^{\alpha j - h - 1}(1+A(1-u))}{1+A(1-u)} \cdot (-A) \right\},
\end{aligned}$$

and we obtain

$$\begin{aligned}
& \frac{d^{k+1}}{du^{k+1}} E_{\nu,1}(-\log^\alpha(1+A(1-u))) \\
&= \frac{(-A)^{k+1}}{(1+A(1-u))^{k+1}} \left\{ -k \sum_{j \geq 0} \sum_{h=1}^k \frac{(-1)^j \Gamma(\alpha j + 1)}{\Gamma(\alpha j - h + 1) \Gamma(\nu j + 1)} s_{k,h} \log^{\alpha j - h}(1+A(1-u)) \right. \\
&\quad \left. + \sum_{j \geq 0} \sum_{h=0}^k \frac{(-1)^j \Gamma(\alpha j + 1)}{\Gamma(\alpha j - h) \Gamma(\nu j + 1)} s_{k,h} \log^{\alpha j - h - 1}(1+A(1-u)) \right\}
\end{aligned}$$

because $\frac{\alpha j - h}{\Gamma(\alpha j - h + 1)} = \frac{1}{\Gamma(\alpha j - h)}$ and $s_{k,0} = 0$; then we get

$$\begin{aligned}
& \frac{d^{k+1}}{du^{k+1}} E_{\nu,1}(-\log^\alpha(1+A(1-u))) \\
&= \frac{(-A)^{k+1}}{(1+A(1-u))^{k+1}} \left\{ -k \sum_{j \geq 0} \sum_{h=1}^k \frac{(-1)^j \Gamma(\alpha j + 1)}{\Gamma(\alpha j - h + 1) \Gamma(\nu j + 1)} s_{k,h} \log^{\alpha j - h}(1+A(1-u)) \right. \\
&\quad \left. + \sum_{j \geq 0} \sum_{m=1}^{k+1} \frac{(-1)^j \Gamma(\alpha j + 1)}{\Gamma(\alpha j - m + 1) \Gamma(\nu j + 1)} s_{k,m-1} \log^{\alpha j - (m-1) - 1}(1+A(1-u)) \right\} \\
&= \frac{(-A)^{k+1}}{(1+A(1-u))^{k+1}} \sum_{j \geq 0} \left\{ \sum_{h=1}^k \frac{(-1)^j \Gamma(\alpha j + 1)}{\Gamma(\alpha j - h + 1) \Gamma(\nu j + 1)} (-k s_{k,h} + s_{k,h-1}) \right. \\
&\quad \left. \cdot \log^{\alpha j - h}(1+A(1-u)) + \frac{(-1)^j \Gamma(\alpha j + 1)}{\Gamma(\alpha j - (k+1) + 1) \Gamma(\nu j + 1)} s_{k,k} \log^{\alpha j - (k+1)}(1+A(1-u)) \right\},
\end{aligned}$$

and (15) holds for $k+1$ because $-k s_{k,h} + s_{k,h-1} = s_{k+1,h}$ and $s_{k,k} = s_{k+1,k+1} = 1$. \square

In view of the next Proposition 3.3 some preliminaries are needed. Firstly let $(O)_\alpha$ be the operator defined by

$$(O)_\alpha f(z) := \begin{cases} \frac{1}{\Gamma(n-\alpha)} \int_{\frac{1-a}{b}}^z \log^{n-1-\alpha} \left(\frac{a+bz}{a+b\tau} \right) \left[\left(\left(\frac{a}{b} + \tau \right) \frac{d}{d\tau} \right)^n f(\tau) \right] \frac{b}{a+b\tau} d\tau & \text{if } \alpha \in (n-1, n) \\ \left(\left(\frac{a}{b} + z \right) \frac{d}{dz} \right)^n f(z) & \text{if } \alpha = n \end{cases} \quad (17)$$

where $z > \frac{1-a}{b}$. Here, for the moment, we are assuming that $\alpha > 0$ and n is an integer value. Thus, for $\alpha \in (n-1, n)$, this operator can be formally considered as the regularized Caputo-like

fractional power of the operator $(\frac{a}{b} + z) \frac{d}{dz}$. Indeed it can be found from the definition of Caputo fractional derivative of order α , by means of the simple transformation $z \mapsto \log(\frac{a}{b} + z)$. Moreover we observe that, if $a = 0$ and $b = 1$, (17) coincides with the Caputo-like regularized Hadamard fractional derivative recently introduced in [10].

In what follows we focalize our attention on the case $\alpha \in (0, 1)$ and, in view of the proof of Proposition 3.3, we check that

$$(O)_\alpha E_{\alpha,1}(-\gamma \log^\alpha(a + bz)) = -\gamma E_{\alpha,1}(-\gamma \log^\alpha(a + bz)). \quad (18)$$

In fact, by (17), for $\beta > -1$ we have

$$\begin{aligned} (O)_\alpha \log^\beta(a + bz) &= \frac{1}{\Gamma(1-\alpha)} \int_{\frac{1-a}{b}}^z \log^{-\alpha} \left(\frac{a+bz}{a+b\tau} \right) \left[\left(\left(\frac{a}{b} + \tau \right) \frac{d}{d\tau} \right) \log^\beta(a + b\tau) \right] \frac{b}{a+b\tau} d\tau \\ &= \frac{1}{\Gamma(1-\alpha)} \int_{\frac{1-a}{b}}^z (\log(a + bz) - \log(a + b\tau))^{-\alpha} \cdot \beta \cdot \frac{\log^{\beta-1}(a + b\tau)}{a + b\tau} b d\tau \end{aligned}$$

and, after some computations with the change of variable $y = \frac{\log(a+b\tau)}{\log(a+bz)}$, we obtain

$$(O)_\alpha \log^\beta(a + bz) = \frac{\beta}{\Gamma(1-\alpha)} \log^{\beta-\alpha}(a + bz) \int_0^1 (1-y)^{-\alpha} y^{\beta-1} dy,$$

and therefore

$$(O)_\alpha \log^\beta(a + bz) = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)} \log^{\beta-\alpha}(a + bz); \quad (19)$$

then, by (19) and some computations, we get

$$(O)_\alpha E_{\alpha,1}(-\gamma \log^\alpha(a + bz)) = \sum_{k=1}^{\infty} \frac{(-\gamma)^k \log^{\alpha k - \alpha}(a + bz)}{\Gamma(\alpha k - \alpha + 1)} = -\gamma \sum_{k=0}^{\infty} \frac{(-\gamma)^k \log^{\alpha k}(a + bz)}{\Gamma(\alpha k + 1)},$$

which meets (18).

Proposition 3.3. Assume that $r = 1$ and let $\{G_{N_\rho^{\nu,\nu}(t)} : t \in [0, T]\}$ be the probability generating functions in Proposition 3.1 with $\alpha = \nu$. Then we have the following results.

(i) (Case $\rho = 1$) Let $(O)_{\nu,1}$ be the operator in (17) with $a = \frac{1}{p}$ and $b = \frac{p-1}{p}$; then

$$(O)_{\nu,1} G_{N_1^{\nu,\nu}(t)}(u) = -G_{N_1^{\nu,\nu}(t)}(u) + 1 - \frac{\frac{1}{q(t)} - 1}{\frac{1}{p} - 1} \quad (\text{for all } u \in (1, 1/(1-p))).$$

(ii) (Case $\rho = 0$) Let $(O)_{\nu,0}$ be the operator in (17) with $a = \frac{1}{q(t)}$ and $b = \frac{q(t)-1}{q(t)}$; then

$$(O)_{\nu,0} G_{N_0^{\nu,\nu}(t)}(u) = -G_{N_0^{\nu,\nu}(t)}(u) \quad (\text{for all } u \in (1, 1/(1-q(t)))).$$

(iii) In both cases (i) and (ii) we have $G_{N_\rho^{\nu,\nu}(t)}(\frac{1-a}{b}) = 1$.

Proof. We start with (i). For $\alpha = \nu \in (0, 1)$ we have

$$\begin{aligned} (O)_{\nu,1} G_{N_1^{\nu,\nu}(t)}(u) &= \frac{\frac{1}{q(t)} - 1}{\frac{1}{p} - 1} (O)_{\nu,1} E_{\nu,1} \left(-\log^\nu \left(\frac{1 - (1-p)u}{p} \right) \right) \\ &= -\frac{\frac{1}{q(t)} - 1}{\frac{1}{p} - 1} E_{\nu,1} \left(-\log^\nu \left(\frac{1 - (1-p)u}{p} \right) \right) = -G_{N_1^{\nu,\nu}(t)}(u) + 1 - \frac{\frac{1}{q(t)} - 1}{\frac{1}{p} - 1} \end{aligned}$$

where (for $\rho = 1$, $a = \frac{1}{p}$, $b = \frac{p-1}{p}$ and $\gamma = 1$) we have used Proposition 3.1 and, for the second equality, (18). Note that we have $u \in (1, 1/(1-p))$ because $G_{N_1^{\nu,\nu}(t)}(u)$ is finite for $|u| < \frac{1}{1-p}$ (see Proposition 3.1 with $\rho = 1$) and $\frac{1-a}{b} = 1$. For $\alpha = \nu = 1$ it is easy to check with some computations that

$$\left(\frac{1}{p-1} + u\right) \frac{d}{du} G_{N_1^{1,1}(t)}(u) = -G_{N_1^{1,1}(t)}(u) + 1 - \frac{\frac{1}{q(t)} - 1}{\frac{1}{p} - 1}.$$

by Proposition 3.1 (in fact we have $\frac{a}{b} = \frac{1}{p-1}$).

We proceed similarly for (ii). For $\alpha = \nu \in (0, 1)$ we have

$$\begin{aligned} (O)_{\nu,0} G_{N_1^{\nu,\nu}(t)}(u) &= (O)_{\nu,0} E_{\nu,1} \left(-\log^\nu \left(\frac{1 - (1-q(t))u}{q(t)} \right) \right) \\ &= -E_{\nu,1} \left(-\log^\nu \left(\frac{1 - (1-q(t))u}{q(t)} \right) \right) = -G_{N_0^{\nu,\nu}(t)}(u) \end{aligned}$$

where (for $\rho = 0$, $a = \frac{1}{q(t)}$, $b = \frac{q(t)-1}{q(t)}$ and $\gamma = 1$) we have used Proposition 3.1 and, for the second equality, (18). Note that we have $u \in (1, 1/(1-p))$ arguing as we did for the proof of (i). For $\alpha = \nu = 1$ it is easy to check with some computations that

$$\left(\frac{1}{q(t)-1} + u\right) \frac{d}{du} G_{N_0^{1,1}(t)}(u) = -G_{N_0^{1,1}(t)}(u)$$

by Proposition 3.1 (in fact we have $\frac{a}{b} = \frac{1}{q(t)-1}$).

Finally (iii) trivially holds because we always have $G_{N_\rho^{\alpha,\nu}(t)}(1) = 1$ (even if $\alpha \neq \nu$) and, in both cases (i) and (ii), $\frac{1-a}{b} = 1$. \square

4 On weighted processes

In this section we consider $\{N_\rho^w(t) : t \in [0, T]\}$ where

$$N_\rho^w(t) := \sum_{n=1}^{M_g^w} 1_{[0,t]}(X_n^{F,\rho})$$

and the probability mass function of the random variable M_g^w is given by

$$P(M_g^w = k) = \frac{P(M_g = k)w(k)}{\mathbb{E}[w(M_g)]} \quad (\text{for all } k \geq 0) \quad (20)$$

for some nonnegative numbers (weights) $\{w(k) : k \geq 0\}$ such that

$$\mathbb{E}[w(M_g)] := \sum_{r=0}^{\infty} w(r)P(M_g = r) \in (0, \infty);$$

then we are referring to the concept of weighted probability mass function (see e.g. [11], p. 90, and the references cited therein).

We remark that M_g^w has the same distribution of M_g if $w(k) = 1$ (for all $k \geq 0$). More in general we have the following well-known property of the weighted probability mass functions: if we consider “proportional weights”

$$\{w(k) : k \geq 0\} \propto \{\tilde{w}(k) : k \geq 0\},$$

i.e. if, for some $c > 0$, we have $w(k) = c\tilde{w}(k)$ (for all $k \geq 0$), then we have the same weighted probability mass function.

The aim of this section is to illustrate the “weighted version structure” for the probability mass function of $N_\rho^w(t)$ for each $t \in (0, T]$, i.e.

$$P(N_\rho^w(t) = k) = \frac{P(N_\rho(t) = k)w(k, t)}{\mathbb{E}[w(N_\rho(t), t)]} \quad (\text{for all } k \geq 0) \quad (21)$$

for some weights $\{w(k, t) : k \geq 0\}$ which depend on $t \in (0, T]$ (obviously we have $w(k, T) = w(k)$ for all $k \geq 0$, i.e. (21) meets (20) when $t = T$). Moreover we give the corrected version of some formulas in [9].

Proposition 4.1. *We set*

$$\begin{aligned} q(k|n, F(t), \rho) &:= (1 - \rho) \binom{n}{k} F^k(t) (1 - F(t))^{n-k} \\ &+ \rho F^{k/n}(t) (1 - F(t))^{1-k/n} 1_{\{0, n\}}(k) \quad (\text{for all } k \in \{0, 1, \dots, n\}). \end{aligned}$$

Then, for all $t \in (0, T]$, we have

$$w(k, t) \propto \frac{\sum_{n=k}^{\infty} q(k|n, F(t), \rho) P(M_g = n) w(n)}{\sum_{n=k}^{\infty} q(k|n, F(t), \rho) P(M_g = n)} \quad (\text{for all } k \geq 0). \quad (22)$$

Proof. By (7) in [9] we have the following generalization of (3):

$$P(N_\rho(t) = k) = \sum_{n=k}^{\infty} q(k|n, F(t), \rho) P(M_g = n) \quad (\text{for all } k \geq 0). \quad (23)$$

Moreover, by (23) (with $N_\rho^w(t)$ and M_g^w in place of $N_\rho(t)$ and M_g) and (20), we obtain

$$P(N_\rho^w(t) = k) = \frac{\sum_{n=k}^{\infty} q(k|n, F(t), \rho) P(M_g = n) w(n)}{\mathbb{E}[w(M_g)]}.$$

Then (21) and the last equality yield

$$\begin{aligned} w(k, t) &= \frac{\mathbb{E}[w(N_\rho(t), t)]}{P(N_\rho(t) = k)} \cdot \frac{\sum_{n=k}^{\infty} q(k|n, F(t), \rho) P(M_g = n) w(n)}{\mathbb{E}[w(M_g)]} \\ &\propto \frac{\sum_{n=k}^{\infty} q(k|n, F(t), \rho) P(M_g = n) w(n)}{P(N_\rho(t) = k)}. \end{aligned}$$

We conclude the proof by taking into account (23) for the denominator in the last expression. \square

Now the correction of (17) and (18) in [9]:

$$\text{Cov}(N_\rho(t), N_\rho(s)) = \lambda s \{1 + \lambda \rho(1 - t)\}$$

and

$$\text{Cov}(N_\rho(t) - N_\rho(s), N_\rho(s)) = -\lambda^2 \rho s(t - s).$$

We also present the corrected version of the displayed formula in Example 4.1 in [9]. We refer to (2) in this note and, in order to have a strict connection with the presentation in [9], we consider $t \in [0, 1]$ in place of $F(t)$ with $t \in [0, T]$. We have to choose

$$P(N_0(t) = k) = \frac{(\lambda t)^k}{k!} e^{-\lambda t} \left(1 - t + \frac{k}{\lambda}\right) \quad (\text{for all } k \geq 0)$$

for the case $\rho = 0$ (see a displayed formula in Section 3.1 in [4]) and

$$P(M_g = k) = \begin{cases} \frac{\lambda^{k-1}}{(k-1)!} e^{-\lambda} & \text{if } k \geq 1 \\ 0 & \text{if } k = 0; \end{cases}$$

then we get

$$\begin{aligned} P(N_\rho(t) = k) &= (1 - \rho) \frac{(\lambda t)^k}{k!} e^{-\lambda t} \left(1 - t + \frac{k}{\lambda} \right) + \rho \left\{ (1 - t) 1_{k=0} + t \cdot \frac{\lambda^{k-1}}{(k-1)!} e^{-\lambda} \cdot 1_{k \geq 1} \right\} \\ &= \begin{cases} (1 - \rho) e^{-\lambda t} (1 - t) + \rho (1 - t) & \text{if } k = 0 \\ (1 - \rho) \frac{(\lambda t)^k}{k!} e^{-\lambda t} \left(1 - t + \frac{k}{\lambda} \right) + \rho t \frac{\lambda^{k-1}}{(k-1)!} e^{-\lambda} & \text{if } k \geq 1, \end{cases} \end{aligned}$$

which is the corrected version of the displayed formula in Example 4.1 in [9].

Acknowledgements. We thank the referee for some useful comments and Federico Polito for Figure 1.

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